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# Dynamic R&D Networks with Process and Product Innovations

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**Abstract**

We analyze the endogenous formation of R&D networks, where firms are active in different product markets and can benefit from R&D spillovers form collaborating firms within or across different industries. R&D spillovers help firms to introduce process innovations to lower their production costs. Product innovations introduce an escape-competition effect, through which firms can enter new markets with fewer competitors. We provide a complete equilibrium characterization in which both, the network of R&D collaborations as well as the market structure, are endogenously determined. We show that the coevolution of market and network structure matter for the relationship between competition and innovation. Moreover, our model allows us to explain differences in the R&D network structures observed across different industries, and how they are related to different levels of competition in these industries.

*Key words:* R&D collaborations, networks, competition, innovation
*JEL:* D85, L24, O33

## 1. Introduction

R&D collaborations have become a widespread phenomenon especially in technologically intensive industries such as, for instance, the pharmaceutical, chemical and computer industries [cf. Ahuja, 2000; Lychagin et al., 2010; Powell et al., 2005; Roijakkers and Hagedoorn, 2006]. In this paper we introduce a tractable (and empirically estimable)[1] model of R&D collaboration networks in which both, the network structure as well as the market structure are endogenous. Our framework allows us to understand the formation of these networks, how they are influenced, and influence competition in different markets. Our model can therefore be used to evaluate and design industrial policy programs to promote welfare in highly dynamic, R&D intensive industries [Bulow et al., 1985; Spencer and Brander, 1983].

A key ingredient of our model is an *escape-competition* effect à la Aghion et al. [2005, 2001]. The escape-competition effect operates at two different levels in our model: At the level of R&D collaborations, firms can escape the competition with their rivals by becoming more productive through reducing their production costs (*process innovations*). A firm can reduce its marginal cost of production not only trough investing into R&D, but also through the formation of R&D collaborations, which allow the firm to benefit from R&D spillovers from its collaboration partners. At the level of markets, firms can escape competition by developing new products and entering new markets where the competitive pressure is lower and profits are higher (*product innovations*) [Aghion and Howitt, 2009]. We investigate how these two forces interact, and when one dominates over the other.

Moreover, we analyze the effect of competition on innovation [Aghion et al., 2005], and show

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[1]Similar estimation algorithms as in Hsie et al. [2012] can be used to estimate the parameters of our model.

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that it depends on the underlying network structure of R&D collaborating firms. In particular, we show that for weakly centralized networks, an increase in competition leads to a decrease in the R&D expenditures per firms, while in strongly centralized networks the opposite holds. While for strongly centralized networks with a dominant firm an increase in competition leads to the exit of many small firms that move to other markets (escape competition effect dominates), in weakly centralized networks an increase in competition does not lead to exit, but reduces overall R&D expenditures per firm as the returns from innovation are declining (Schumpeterian effect dominates).

Further, our approach allows us to explain the differences in the R&D network structures observed across different industries [Rosenkopf and Schilling, 2007]. We investigate how competition affects the formation of R&D collaborations, as well as the equilibrium market structure. In particular, we analyze how the different R&D collaboration intensities across different industries – such as the biotech sector [Aharonson et al., 2008; Powell et al., 2005] compared to other sectors [Hagedoorn, 2002; Schilling, 2009] – can be explained from different levels of competition in these industries. Moreover, we show that our model can explain the core-periphery structure observed in real-world R&D networks [Kitsak et al., 2010], and the differences in the concentration and hierarchical structure of collaborations across various sectors.[2] Further, we analyze how the escape-competition effect shapes the equilibrium distribution of firms across sectors and affects market entry. This can help us to understand the impact of different market sizes [Acemoglu et al., 2006; Acemoglu and Linn, 2004; Dubois et al., 2011], competitive pressure, or differences in the R&D intensity across sectors [Ang, 2008; Rosenkopf and Schilling, 2007], on R&D collaboration networks.

The introduction of the escape-competition effect also has important welfare implications. Whereas in previous models such as König et al. [2014b] an increase in competition always leads to a reduction in output (cf. *Schumpeterian effect*, see e.g. Aghion et al. [2014]), R&D and welfare, here firms can escape competition by entering niche markets where they can expand their production, and perform higher levels of R&D. Our analysis thus provides new insights into the relationship between innovation and competition in R&D intensive industries in which R&D collaborations are commonly observed. Our model can further be used to identify the optimal network and/or market structure, and evaluate potential inefficiencies that arise in the decentralized equilibrium due to the presence of network externalities.

The paper is organized as follows. Firms' profits and the spillovers from collaborations are introduced in Section 2. Section 3 analyzes the equilibrium production levels in an exogenous network. Section 4 then introduces the industry and network dynamics and provides a complete equilibrium characterization. Further, Section 5 introduces an extension of the model with multiproduct firms. Finally, Section 6 concludes. Appendix A provides additional details to some network definitions used in the paper, while all proofs can be found in Appendix B.

## 2. Firms' Profits and Production

Firms can reduce their production costs by investing in R&D as well as by establishing an R&D collaboration with another firm. The amount of this cost reduction depends on the R&D effort $e_i$ of firm $i$ and the R&D efforts of the firms that are collaborating with $i$, i.e., R&D collaboration partners.[3] Given the effort level $e_i \in \mathbb{R}_+$, the marginal cost $c_i$ of firm $i$ is given

[2] E.g. Kitsak et al. [2010] observe that the life sciences industry network consists of a strong core periphery structure, while this structure is less pronounced in the information and communication technology sectors.

[3] See also Kamien et al. [1992] for a similar model in which firms unilaterally choose their R&D effort levels.

by[4]

$$c_i = c_i - e_i - \varphi \sum_{j=1}^{n} a_{ij} e_j, \tag{1}$$

The network $G$ is captured by $\mathbf{A}$, which is a symmetric $n \times n$ *adjacency matrix*. Its element $a_{ij} \in \{0,1\}$ indicates if there exists a link between nodes $i$ and $j$ such that $a_{ij} = 1$ if there is a link $(i,j)$ and zero otherwise. In the context of our model, $a_{ij} = 1$ if firms $i$ and $j$ set up an R&D collaboration (0 otherwise) and $a_{ii} = 0$. In Equation (1), the total cost reduction for firm $i$ stems from its own research effort $e_i$ and the research knowledge of other firms, i.e., *knowledge spillovers*, which is captured by the term $\sum_{j=1}^{n} a_{ij} e_j$, where $\varphi \geq 0$ is the marginal cost reduction due to neighbor's effort. We assume that R&D effort is costly. In particular, the cost of R&D effort is an increasing function, exhibits decreasing returns, and is given by $\frac{1}{2} e_i^2$. Firm $i$'s profit is then given by

$$\pi_i = (p_i - c_i) q_i - \frac{1}{2} e_i^2 - \zeta d_i - \gamma_i, \tag{2}$$

where $\zeta d_i$ and $\gamma_i$ are fixed costs for firm $i$ (see below), and the inverse demand for the good $q_i$ produced by firm $i$ in market $\mathcal{M}_m$, $m = 1, \ldots, M$ is given by

$$p_i = \alpha_i - q_i - \rho \sum_{j \in \mathcal{M}_m, j \neq i} q_j. \tag{3}$$

Inserting marginal cost from Equation (1) and inverse demand from Equation (3) into Equation (18) gives

$$\begin{aligned} \pi_i &= (\bar{\alpha}_i - q_i - \rho \sum_{j \in \mathcal{M}_m, j \neq i} q_j - c_i + e_i + \varphi \sum_{j=1}^{n} a_{ij} e_j) q_i - \frac{1}{2} e_i^2 - \zeta d_i - \gamma_i \\ &= (\bar{\alpha}_i - c_i) q_i - q_i^2 - \rho \sum_{j=1}^{n} b_{ij} q_i q_j + q_i e_i + \varphi q_i \sum_{j=1}^{n} a_{ij} e_j - \frac{1}{2} e_i^2 - \zeta d_i - \gamma_i, \end{aligned} \tag{4}$$

where $b_{ij} \in \{0,1\}$ indicates whether firms $i$ and $j$ operate in the same market or not, and let $\mathbf{B}$ be the $n \times n$ matrix whose $ij$-th element is $b_{ij}$. In Equation (4), we have that $\sum_{j \in \mathcal{M}_m, j \neq i} q_j = \sum_{j=1}^{n} b_{ij} q_j$ since $b_{ij} = 1$ if $i, j \in \mathcal{M}_m$ and $i \neq j$, and $b_{ij} = 0$ otherwise, i.e. if $i$ and $j$ do not belong to the same market. In other words, the matrix $\mathbf{B}$ captures which firms operate in the same market and which firms do not. Take row $i$ in matrix $\mathbf{B}$, for example. If there are only zeros, this means that firm $i$ is alone in its market. If there is a 1 corresponding to column $j$, this means that firms $i$ and $j$ operate in the same market (or sector). Note that the matrix $\mathbf{B}$ can always be represented in block diagonal form. We denote by $\mathcal{B}(n,m)$ the class of zero-one block diagonal $n \times n$ matrices with $m$ blocks, so that $\mathbf{B} \in \mathcal{B}(n,m)$.

The FOC of profits with respect to R&D effort $e_i$ of firm $i$ is given by

$$\frac{\partial \pi_i}{\partial e_i} = q_i - e_i = 0,$$

[4] This generalizes earlier studies such as that by D'Aspremont and Jacquemin [1988] where spillovers are assumed to take place between all firms in the industry and no distinction between collaborating and non-collaborating firms is made.

so that we obtain

$$e_i = q_i. \tag{5}$$

Inserting the relationship $e_i = q_i$ and denoting by $\eta_i \equiv \bar{\alpha}_i - c_i$, we can write Equation (4) as [cf. Ballester et al., 2006]

$$\pi_i = \underbrace{\eta_i q_i - q_i^2}_{\text{own concavity}} \underbrace{-\rho \sum_{j \neq i}^{n} b_{ij} q_i q_j}_{\text{global substitutability}} + \underbrace{\varphi \sum_{j=1}^{n} a_{ij} q_i q_j}_{\text{local complementarity}} \underbrace{-\zeta d_i - \gamma_i}_{\text{fixed cost}}. \tag{6}$$

We assume that the fixed cost can be decomposed in a network dependent part, $\zeta d_i$ and an industry/market dependent part, $\gamma_i$. The network dependent part captures collaboration costs, where $d_i$ counts the number of collaborations of firm $i$. The industry/market dependent cost $\gamma_i = \gamma^m \mathbb{1}_{\{i \in \mathcal{M}_m\}}$ captures fixed market entry costs and barriers to entry. Indeed, $\gamma^m$ is the entry cost to market $m$. We then can write $\zeta_i + \gamma_i = \zeta d_i + \sum_{m=1}^{M} \gamma^m \mathbb{1}_{\{i \in \mathcal{M}_m\}}$, and firm $i$'s profit can be written as

$$\pi_i = \eta_i q_i - q_i^2 - \rho \sum_{j \neq i}^{n} b_{ij} q_i q_j + \varphi \sum_{j=1}^{n} a_{ij} q_i q_j - \zeta d_i - \sum_{m=1}^{M} \gamma^m \mathbb{1}_{\{i \in \mathcal{M}_m\}}. \tag{7}$$

Note that the profit function introduced in Equation (7) admits a potential function [cf. Monderer and Shapley, 1996], which not only accounts for quantity adjustments but also for the linking and market entry strategies.

**Proposition 1.** *Then the profit function of Equation (7) admits a potential game with potential function* $\Phi \colon \mathbb{R}_+^n \times \{0,1\}^{n \times n} \times \mathcal{B}(n,m) \to \mathbb{R}$ *given by*

$$\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} (\eta q_i - \nu q_i^2) - \frac{\rho}{2} \sum_{i=1}^{n} \sum_{j \neq i} q_i q_j + \frac{\varphi}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_i q_j - \zeta m - \sum_{i=1}^{n} \sum_{m=1}^{M} \gamma^m \mathbb{1}_{\{i \in \mathcal{M}_m\}}, \tag{8}$$

*where $m$ is the number of links in $G$.*

The Nash equilibria are then the states maximizing the potential. The potential function will be useful when analyzing the stationary states of a dynamic process in which the output levels of the firms, their collaborations and the market structure are endogenous as discussed in Appendix 4 (see in particular Proposition 2).

Note that in the absence of collaborations, our framework can be interpreted as a public goods game in which the effort levels of linked agents in a network are strategic substitutes. This generalizes the models analyzed in Bramoullé and Kranton [2007]; Bramoullé et al. [2014] to an endogenous network.

## 3. (Quasi) Equilibrium Characterization

For simplicity we consider only the case of firms with a single product. Moreover, in the following we assume that the network $G$ changes only slowly relative to the rate at which firms adjust their production levels. We then can assume that firms play a (quasi) Nash equilibrium in their output levels for a given network $G$. The first order conditions of Equation (4) with respect to output $q_i$ are given by

$$\frac{\partial \pi_i}{\partial q_i} = \eta_i - 2q_i - \rho \sum_{j=1}^{n} b_{ij} q_j + e_i + \varphi \sum_{j=1}^{n} a_{ij} e_j = 0.$$

Next, inserting equilibrium effort from Equation (5) and rearranging terms gives

$$q_i = \eta_i - \rho \sum_{j=1}^{n} b_{ij} q_j + \varphi \sum_{j=1}^{n} a_{ij} q_j. \tag{9}$$

In matrix-vector notation this can be written as

$$\mathbf{q} = \boldsymbol{\eta} - \rho \mathbf{B} \mathbf{q} + \varphi \mathbf{A} \mathbf{q},$$

or, equivalently,

$$(\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A}) \mathbf{q} = \boldsymbol{\eta}.$$

When the matrix $\mathbf{I}_n - \rho \mathbf{B} + \varphi \mathbf{A}$ is invertible,[5] equilibrium quantities can be explicitly computed as

$$\mathbf{q} = (\mathbf{I}_n - \rho \mathbf{B} + \varphi \mathbf{A})^{-1} \boldsymbol{\eta}. \tag{10}$$

Further, one can show that equilibrium profits are given by $\pi_i = \frac{1}{2} q_i^2$. In the special case of a single market (i.e., $M = 1$) and homogeneous firms, $\eta_i = \eta$ for all $i = 1, \ldots, n$, the Nash equilibrium output levels from Equation (10) can be simplified to

$$\mathbf{q} = \frac{\eta}{1 - \rho + \rho \|\mathbf{b}_{\mathbf{u}}(G, \varphi)\|_1} \mathbf{b}(G, \varphi). \tag{11}$$

The vector $\mathbf{b}(G, \varphi) \equiv (\mathbf{I}_n - \varphi \mathbf{A})^{-1} \mathbf{u}$ ,with $\mathbf{u}$ being a vector of ones, in Equation (11) corresponds to a well known measure for the centrality of the nodes in a network called the *Katz-Bonacich* centrality [Bonacich, 1987]. More details and a precise definition of this centrality measure can be found in Appendix A.

## 4. Industry and Network Dynamics

In this section we introduce a stochastic dynamic process in which both, the network structure and the market structure are endogenously formed, based on the profit maximizing decisions of firms with whom to collaborate, or which market to enter. The opportunities for change arrive as a Poisson process [cf. Sandholm, 2010], similar to Calvo models of pricing [Calvo, 1983]. To capture the fact that the establishment of an R&D collaboration is fraught with ambiguity and uncertainty [cf. Czarnitzki et al., 2015; Czarnitzki and Toole, 2013; Kelly et al., 2002], and there is uncertainty in the profitability of market entry, we will introduce noise in this decision process. The precise definition of the dynamics of quantity adjustment, network evolution and endogenous competition structure is given in the following:

**Definition 1.** *The evolution of the population of firms, the collaborations between them and the market structure are characterized by a sequence of states $(\boldsymbol{\omega}_t)_{t \in \mathbb{R}_+}$, $\omega_t \in \Omega$, where each state $\boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)$ consists of a vector of firms' output levels $\mathbf{q}_t \in \mathcal{Q}^n$, a network of collaborations represented by the adjacency matrix $\mathbf{A}_t \in \{0,1\}^{n \times n}$, where $a_{ij,t} = 1$ if firms $i$ and $j$ collaborate,*

[5] For sufficient conditions guaranteeing invertibility see König et al. [2014b].

*and* $a_{ij,t} = 0$ *otherwise, and a competition matrix* $\mathbf{B}_t \in \mathcal{B}(n,m)$,[6] *where* $b_{ij,t} = 1$ *if firms* $i$ *and* $j$ *operate in the same market at time* $t$ *and* $b_{ij,t} = 0$ *otherwise. We assume that firms choose quantities from a bounded set* $\mathcal{Q}$. *In a short time interval* $[t, t+\Delta t)$, $t \in \mathbb{R}_+$, *one of the following events happens:*

**Output adjustment** *At rate* $\chi \geq 0$ *a firm* $i \in \mathcal{N}$ *is selected at random and given a revision opportunity of its current output. When firm* $i$ *receives such a revision opportunity, the probability to choose the output is governed by a multinomial logistic function with choice set* $\mathcal{Q}$ *and parameter* $\vartheta$ *[cf. Anderson et al., 2001; McFadden, 1976], so that the probability that we observe a switch by firm* $i$ *to an output level in the infinitesimal interval* $[q, q+dq]$ *is given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (q \leq q_i \leq q+dq, \mathbf{q}_{-it}, \mathbf{A}_t, \mathbf{B}_t) | \boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \\ = \chi \frac{e^{\vartheta \pi_i(q, \mathbf{q}_{-it}, \mathbf{A}_t, \mathbf{B}_t)} dq}{\int_{\mathcal{Q}} e^{\vartheta \pi_i(q', \mathbf{q}_{-it}, \mathbf{A}_t, \mathbf{B}_t)} dq'} \Delta t + o(\Delta t). \quad (12)\end{aligned}$$

**Link formation** *With rate* $\lambda > 0$ *a pair of firms* $ij$ *which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal payoff the firms receive from the link plus an additive pairwise i.i.d. error term* $\varepsilon_{ij,t}$. *The probability that link* $ij$ *is created is then given by*

$$\begin{aligned}&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) | \boldsymbol{\omega}_{t-1} = (\mathbf{q}, \mathbf{A}_t, \mathbf{B}_t)\right) \\ &= \lambda\, \mathbb{P}\left(\{\pi_i(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) + \varepsilon_{ij,t} > 0\}\right. \\ &\quad \left.\cap \{\pi_j(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \pi_j(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) + \varepsilon_{ij,t} > 0\}\right) \Delta t + o(\Delta t),\end{aligned}$$

*where we have denoted by* $\mathbf{A}_t + ij$ *the adjacency matrix obtained from* $\mathbf{A}_t$ *by adding the link* $ij$. *Using the fact that* $\pi_i(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) = \pi_j(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \pi_j(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) = \Phi(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)$, *and assuming that the error term is i.i. logistically distributed,*[7] *we obtain for the creation of the link* $ij$

$$\begin{aligned}&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) | \boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \\ &= \lambda\, \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t) - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \Delta t + o(\Delta t) \\ &= \lambda \frac{e^{\vartheta \Phi(\mathbf{q}_t, \mathbf{A}_t + ij, \mathbf{B}_t)}}{e^{\vartheta \Phi(\mathbf{q}_t, \mathbf{A}_t + ij)} + e^{\vartheta \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)}} \Delta t + o(\Delta t), \quad (13)\end{aligned}$$

**Link removal** *With rate* $\xi > 0$ *a pair of connected firms* $ij$ *receives an opportunity to terminate their connection. The link is removed if at least one firm finds this profitable. The marginal payoffs from removing the link* $ij$ *are perturbed by an additive pairwise i.i.d. error term*

[6] Recall that $\mathcal{B}(n,m)$ denotes the class of zero-one block diagonal $n \times n$ matrices with $m$ blocks and zero diagonal, so that $\mathbf{B}_t \in \mathcal{B}(n,m)$ for all $t \geq 0$.

[7] Let $z$ be i.i. logistically distributed with mean 0 and scale parameter $\vartheta$, i.e. $F_z(x) = \frac{e^{\vartheta x}}{1+e^{\vartheta x}}$. Consider the random variable $\varepsilon = g(z) = -z$. Since $g$ is monotonic decreasing, and $z$ is a continuous random variable, the distribution of $\varepsilon$ is given by $F_\varepsilon(y) = 1 - F_z(g^{-1}(y)) = \frac{e^{\vartheta y}}{1+e^{\vartheta y}}$.

$\varepsilon_{ij,t}$. *The probability that the link* $ij$ *is removed is then given by*

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t)|\boldsymbol{\omega}_t = (\mathbf{q}, \mathbf{A}_t, \mathbf{B}_t)\right) \\
&= \xi\, \mathbb{P}\left(\{\pi_i(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) + \varepsilon_{ij,t} > 0\}\right. \\
&\quad \left.\cup\{\pi_j(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \pi_j(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) + \varepsilon_{ij,t} > 0\}\right) \Delta t + o(\Delta t),
\end{aligned}$$

*where we have denoted by* $\mathbf{A}_t - ij$ *the adjacency matrix obtained from* $\mathbf{A}_t$ *by removing the link* $ij$. *Using the fact that* $\pi_i(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) = \pi_j(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \pi_j(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) = \Phi(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)$, *and assuming that the error term is i.i. logistically distributed we obtain*

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t)|\boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \\
&= \xi\, \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t) - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \Delta t + o(\Delta t) \\
&= \xi \frac{e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t - ij, \mathbf{B}_t)}}{e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t - ij)} + e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)}} \Delta t + o(\Delta t).
\end{aligned}$$

**Product innovation** *With rate* $\varkappa > 0$ *a firm makes a product innovation. When a firm has made a product innovation it adopts the new product (and enters a new market) only if its marginal profits plus an additive i.i.d. error term are positive. The probability that firm* $i$ *enters a new market and the competition structure changes from* $\mathbf{B}_t$ *to* $\mathbf{B}'$ *is then given by*[8]

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}')|\boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \\
&\qquad = \varkappa \mathbb{P}\left(\pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}') - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) + \varepsilon_{it} > 0\right) \Delta t + o(\Delta t).
\end{aligned}$$

*Using the fact that* $\pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}') - \pi_i(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t) = \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}') - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)$, *and assuming that the error term is i.i. logistically distributed we obtain*

$$\begin{aligned}
\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}')|\boldsymbol{\omega}_t = (\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) &= \kappa\, \mathbb{P}\left(-\varepsilon_{i,t} < \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}') - \Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)\right) \Delta t + o(\Delta t) \\
&= \kappa \frac{e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}')}}{e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}')} + e^{\vartheta\Phi(\mathbf{q}_t, \mathbf{A}_t, \mathbf{B}_t)}} \Delta t + o(\Delta t).
\end{aligned}$$

In the above definition we assume that product innovation follows a Poisson process with parameter $\varkappa$ [cf. Aghion and Howitt, 1992, 2009; Aghion et al., 1998]. The probability that a firm successfully innovates and discovers a new product in a short time interval is then proportional to $\varkappa$. However, a firm only adopts a new product when its profits with the new product exceed its profits with the old product, and this depends on the degree of competition in the different markets. Moreover, observe that product innovation incorporates an *escape-competition effect* [cf. Aghion et al., 2005, 2001; Aghion and Howitt, 2009], where firms can enter new markets with fewer competitors. The resulting endogeneity of the competition matrix $\mathbf{B}$ extends the analysis in König et al. [2014b], where an increase in competition always leads to a welfare reduction by lowering the output of firms, while here an increase in competition can lead firms to enter niche markets where they can produce more than when the market structure is fixed.

[8]When a firm $i$ makes a product innovation, it can enter a new market and a transition occurs from $b_{ij,t} = 0$ to $b_{ij,t+\Delta t} = 1$ for all firms $j$ currently active in that new market. We assume that if a firm adopts a new product it abandons the previous product, so that $b_{ik,t} = 1$ to $b_{ik,t+\Delta t} = 0$ for all firms $k$ that were competitors in the previous market.

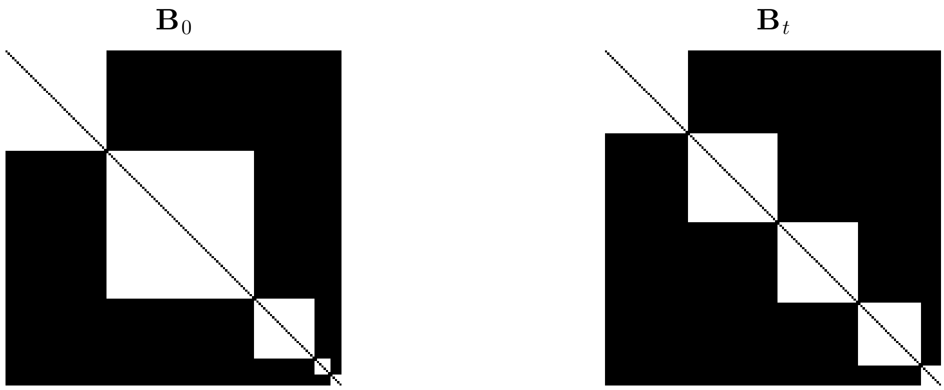


Figure 1: Stationary market structure: (Left panel) The initial block-diagonal competition matrix $\mathbf{B}_0$ in which firms are randomly assigned to sectors with increasing weights across sectors. (Right panel) The stationary competition matrix $\mathbf{B}_t$ exhibiting a uniform distribution of firms across sectors. This is consistent with the theoretical prediction of Proposition 3, where it is shown that in the symmetric case the stochastically stable state is characterized by a homogeneous market structure in which the number of firms across sectors is equalized.

With the potential function $\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B})$ of Proposition 1 we then can state the following proposition.

**Proposition 2.** *The dynamic process $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$ introduced in Definition 1 induces an irreducible and aperiodic Markov chain with a unique stationary distribution $\mu^{\vartheta}\colon \mathcal{Q}^n \times \{0,1\}^{n\times n} \times \mathcal{B}(n,m) \to [0,1]$ such that $\lim_{t\to\infty} \mathbb{P}(\boldsymbol{\omega}_t = (\mathbf{q}, \mathbf{A}, \mathbf{B})|\boldsymbol{\omega}_0 = (\mathbf{q}_0, \mathbf{A}_0, \mathbf{B}_0)) = \mu^{\vartheta}(\mathbf{q}, \mathbf{A}, \mathbf{B})$. When the shocks $\varepsilon$ are independent and identically exponentially distributed with parameter $\vartheta \geq 0$, then the probability measure $\mu^{\vartheta}$ is given by*

$$\mu^{\vartheta}(\mathbf{q}, \mathbf{A}, \mathbf{B}) = \frac{e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) - m\ln\left(\frac{\xi}{\lambda}\right))}}{\sum_{\mathbf{A}'\in\{0,1\}^{n\times n}} \sum_{\mathbf{B}'\in\mathcal{B}(n,m)} \sum_{\mathbf{q}'\in\mathcal{Q}^n} e^{\vartheta(\Phi(\mathbf{q}',\mathbf{A}',\mathbf{B}') - m'\ln\left(\frac{\xi}{\lambda}\right))}}, \tag{14}$$

*with the potential function $\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B})$ given in Equation (8).*

The next proposition characterizes the networks in the support of the stationary distribution as having a core periphery structure similar to what we find in the data [see also Kitsak et al., 2010; Rosenkopf and Schilling, 2007]. In particular, it shows that with decreasing values of the noise parameterized by $\vartheta$ the stochastically stable networks are nested split graphs [König et al., 2014a].

**Proposition 3.** *The states in the support of the stationary distribution $\mu^{\vartheta}$ of Proposition 2 have the following properties:*

*(i) The probability of observing a network $G \in \mathcal{G}_n$, respectively, an adjacency matrix $\mathbf{A}$, given an output distribution $\mathbf{q} \in \mathcal{Q}^n$ and market structure $\mathbf{B} \in \mathcal{B}(n,m)$ is determined by the conditional distribution*

$$\mu^{\vartheta}(\mathbf{A}|\mathbf{q}, \mathbf{B}) = \frac{\mu^{\vartheta}(\mathbf{q}, \mathbf{A}, \mathbf{B})}{\mu^{\vartheta}(\mathbf{q}, \mathbf{B})} = \prod_{i<j}^{n} \frac{e^{\vartheta a_{ij}(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}, \tag{15}$$

*which is equivalent to the probability of observing an inhomogeneous random graph with link probability*

$$p^{\vartheta}(q_i, q_j) = \frac{e^{\vartheta(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}. \tag{16}$$

*(ii) Moreover, in the limit of $\vartheta \to \infty$, the stochastically stable network $G$ with $\mu^*(\mathbf{q}, \mathbf{A}, \mathbf{B}) > 0$ is a nested split graph in which firms $i$ and $j$ are linked if and only if $\rho q_i q_j > \zeta$.*

*(iii) Further, in the stochastically stable state, as $\vartheta \to \infty$, we obtain a homogeneous market structure in which the number of firms across sectors is equalized.*

Typical networks in the support of the stationary distribution with varying values of $\rho \in \{0.25, 0.5, 0.75, 1\}$ are shown Figure 2 assuming a single market. We observe that with increasing values of $\rho$ the network density decreases and networks become more centralized on a few hub firms. That is, with increasing levels of competition network centralization increases. This is consistent with the core-periphery structure predicted by Proposition 3.

Next, we consider the case of multiple sectors. Figure 1 shows the transition of the market structure from an initial block-diagonal competition matrix in which firms are randomly assigned to sectors with increasing weights across sectors to the stationary competition matrix exhibiting a uniform distribution of firms across sectors. This shows that in the homogeneous case, where firms are identical and the degree of substitutability between products ($\rho$) is the same across sectors, the potential maximizing configuration is the one in which all markets have the same size. This is consistent with the theoretical prediction of Proposition 3, where it is shown that in the symmetric case the stochastically stable state is characterized by a homogeneous market structure in which the number of firms across sectors is equalized.

As an alternative example with heterogeneous levels of competition in each sector (i.e. different values of $\rho_m$ across sectors indexed by $m$) consider a situation with only two markets, $\mathcal{M}_1$ and $\mathcal{M}_2$. Figure 3 shows the asymptotic average output $\bar{q}$ produced together with the fraction of firms in market one for increasing values of the competition parameter $\rho$ and varying values of the noise parameter $\vartheta$ with $n = 15$ firms, $\eta = 15$ and $\phi = \nu = \lambda = \xi = \chi = 1$. The underlying economy consists of two sectors, $\mathcal{M}_1$ and $\mathcal{M}_1$, where the degree of substitutability $\rho$ in $\mathcal{M}_1$ is gradually decreased from one to zero, while in $\mathcal{M}_2$ it is fixed at one. We observe that $\bar{q}$ is increasing with decreasing $\rho$, but this effect is much higher when firms can escape competition in the second sector by entering the first sector.

A similar escape competition effect is illustrated in Figure 4. It illustrates the initial and final market structure with a heterogeneous distribution of $\rho$ across 10 sectors gradually increasing over the range $\rho \in \{0.0182, 0.0364, 0.0545, 0.0727, 0.0909, 0.1091, 0.1273, 0.1455, 0.1636, 0.1818\}$. While initially firms are homogeneously distributed across sectors, over time they migrate to sectors with lower $\rho$, that is, less competition.

# 5. An Extension with Multi-Product Firms

In this section we allow firms to produce more than one product and thus be active in different markets [cf. e.g. Bernard et al., 2011; Bulow et al., 1985]. Let $q_{ik} \in \mathbb{R}_+^{nm}$ be the quantity sold by firm $i$ to market $\mathcal{M}_k$. Then the price in market $\mathcal{M}_k$ is given by

$$p_k = \alpha_k - q_{ik} - \rho \sum_{j \in \mathcal{M}_k \setminus \{i\}} q_{jk}.$$

Let $e_{ik}$ be the R&D effort of firm $i$ for products sold to market $\mathcal{M}_k$. Then the marginal cost of producing in market $\mathcal{M}_k$ is given by

$$c_{ik} = \bar{c}_{ik} - e_{ik} - \varphi \sum_{j=1}^{n} a_{ij} e_{jk} \mathbb{1}_{\{j \in \mathcal{M}_k\}}, \tag{17}$$

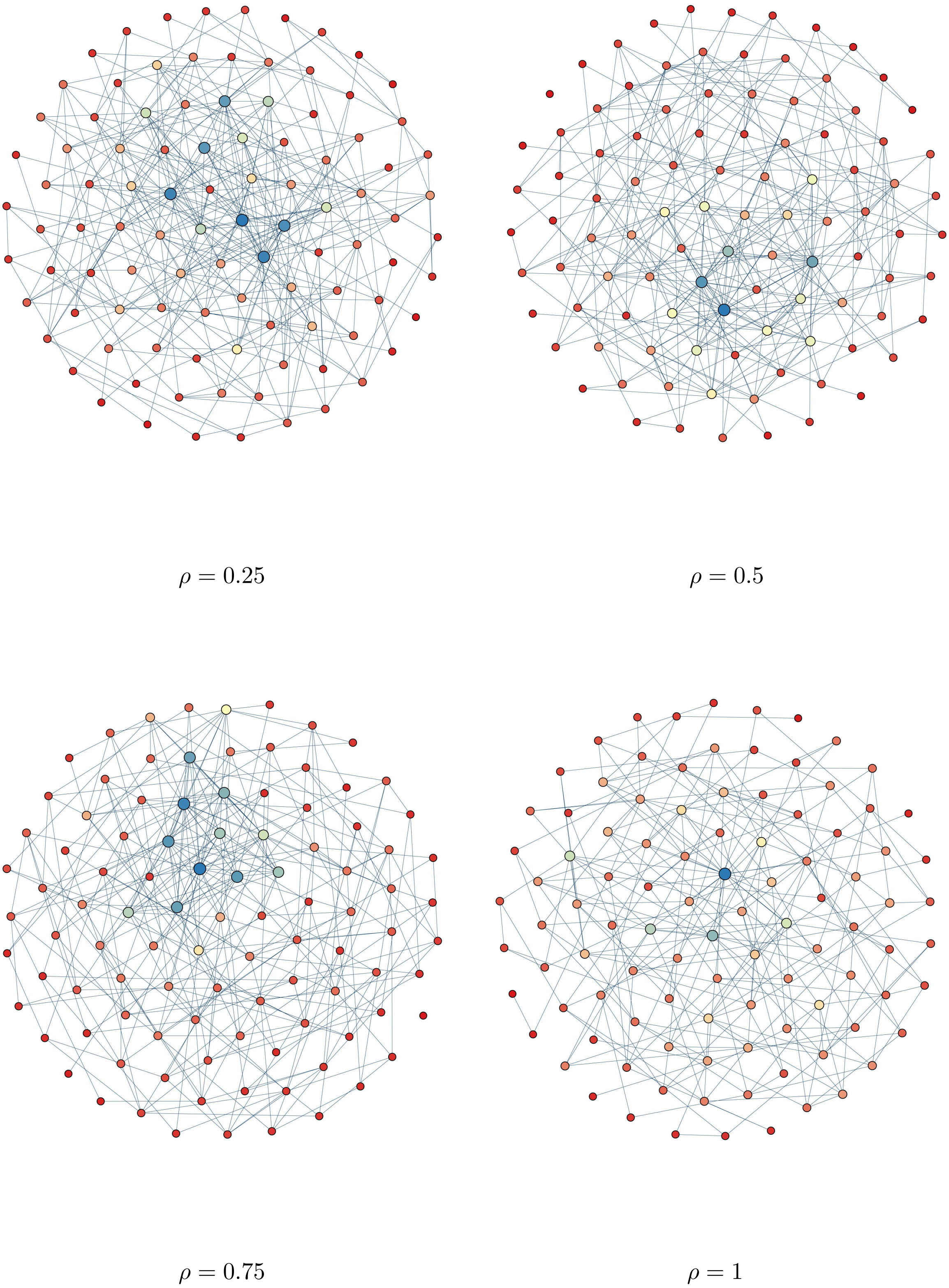


Figure 2: Typical networks in the support of the stationary distribution for $\vartheta = 0.1$ and varying values of $\rho \in \{0.25, 0.5, 0.75, 1\}$. We observe that with increasing values of $\rho$ the network density decreases and networks become more centralized on a few hub firms. This is consistent with the core-periphery structure predicted by Proposition 3.

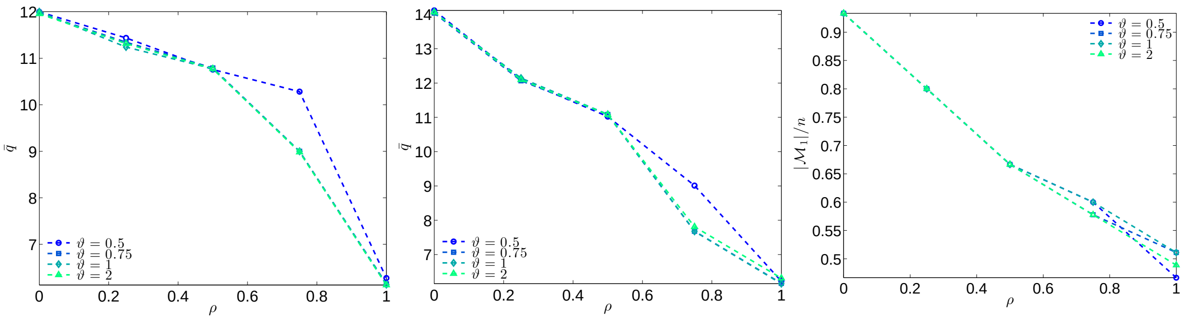


Figure 3: Escape competition effect: Asymptotic average output $\bar{q}$ produced (left and middle panel) together with the fraction of firms in market one (right panel) for increasing values of the competition parameter $\rho$ and varying values of the noise parameter $\vartheta$ with $n = 15$ firms, $\eta = 15$ and $\phi = \nu = \lambda = \xi = \chi = 1$. The underlying economy consists of two sectors, $\mathcal{M}_1$ and $\mathcal{M}_1$, where the degree of substitutability $\rho$ in $\mathcal{M}_1$ is gradually decreased from one to zero, while in $\mathcal{M}_2$ it is fixed at one. The left panel shows the asymptotic value of $\bar{q}$ when firms cannot change sectors (setting $\kappa = 0$), while the middle and right panels allow for market entry ($\kappa = 10$). We observe that $\bar{q}$ is increasing with decreasing $\rho$, but this effect is much higher when firms can escape competition in the second sector by entering the first sector (see the middle and right panel).

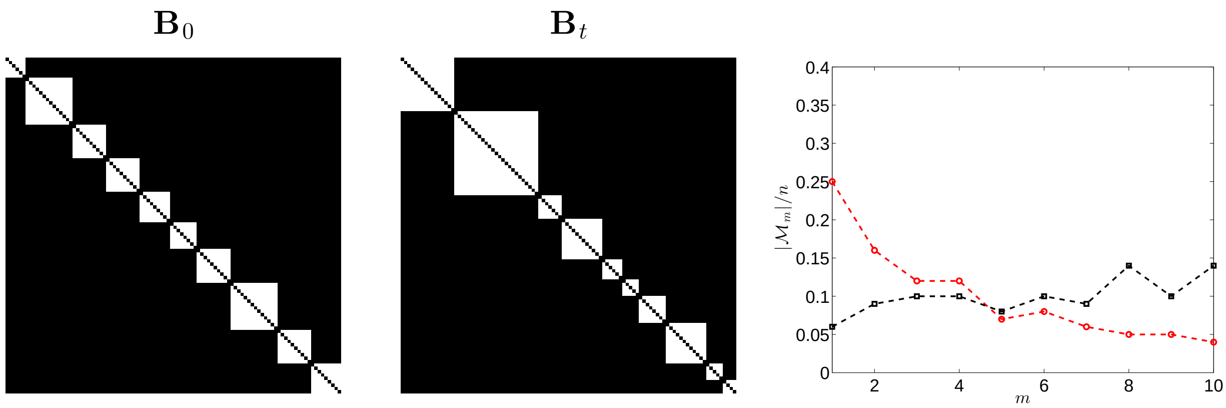


Figure 4: Escape competition effect: The initial and final market structure, represented by $\mathbf{B}_0$ and $\mathbf{B}_t$, respectively, for a heterogeneous distribution of $\rho$ across 10 sectors gradually increasing over the range $\rho \in \{0.0182, 0.0364, 0.0545, 0.0727, 0.0909, 0.1091, 0.1273, 0.1455, 0.1636, 0.1818\}$. While initially firms are homogeneously distributed across sectors, over time they migrate to sectors with lower $\rho$, that is, less competition.

The profit of firm $i$ is given by

$$\pi_i = \sum_{k=1}^{M} \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (p_k - c_{ik})q_{ik} - \frac{1}{2}e_{ik}^2 - \gamma_{ik} \right) - \zeta_i, \tag{18}$$

where $\zeta_i$ and $\gamma_{ik}$ are fixed costs for firm $i$ (see below). Inserting inverse demand and marginal cost yields

$$\begin{aligned}
\pi_i &= \sum_{k=1}^{M} \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( \left( \alpha_k - q_{ik} - \rho \sum_{j\in\mathcal{M}_m, j\neq i} q_{jk} - \bar{c}_{ik} + e_{ik} + \varphi \sum_{j=1}^{n} a_{ij} e_{jk} \mathbb{1}_{\{j\in\mathcal{M}_k\}} \right) q_{ik} \right.\\
&\quad \left. - \frac{1}{2}e_{ik}^2 - \gamma_{ik} \right) - \zeta d_i \\
&= \sum_{k=1}^{M} \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - q_{ik}^2 - \rho \sum_{j\in\mathcal{M}_m, j\neq i} q_{jk}q_{ik} + e_{ik}q_{ik} + \varphi \sum_{j=1}^{n} a_{ij} e_{jk} \mathbb{1}_{\{j\in\mathcal{M}_k\}} q_{ik} \right.\\
&\quad \left. - \frac{1}{2}e_{ik}^2 - \gamma_{ik} \right) - \zeta d_i.
\end{aligned} \tag{19}$$

The FOC w.r.t. $e_{ik}$ is given by

$$\frac{\partial \pi_i}{\partial e_{ik}} = \mathbb{1}_{\{i\in\mathcal{M}_k\}} (q_{ik} - e_{ik}) = 0,$$

so that when $i \in \mathcal{M}_k$ we obtain

$$e_{ik} = q_{ik}. \tag{20}$$

Inserting into profits yields

$$\begin{aligned}
\pi_i &= \sum_{k=1}^{M} \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - \frac{1}{2}q_{ik}^2 - \gamma_{ik} \right) - \rho \sum_{k=1}^{M} \sum_{j\neq i}^{n} \mathbb{1}_{\{i,j\in\mathcal{M}_k\}} q_{ik}q_{jk} \\
&\quad + \varphi \sum_{k=1}^{M} \sum_{j=1}^{n} a_{ij} \mathbb{1}_{\{i,j\in\mathcal{M}_k\}} q_{ik}q_{jk} - \zeta d_i \\
&= \sum_{k=1}^{M} \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - \frac{1}{2}q_{ik}^2 - \gamma_{ik} \right) - \rho \sum_{k=1}^{M} \sum_{j\neq i}^{n} b_{ij,k} q_{ik}q_{jk} \\
&\quad + \varphi \sum_{k=1}^{M} \sum_{j=1}^{n} a_{ij,k} q_{ik}q_{jk} - \zeta d_i,
\end{aligned} \tag{21}$$

where $b_{ij,k} = \mathbb{1}_{\{i\in\mathcal{M}_k\}}\mathbb{1}_{\{j\in\mathcal{M}_k\}} \in \{0,1\}$ indicates whether both firms $i$ and $j$ are active in market $\mathcal{M}_k$ and $a_{ij,k} = a_{ij}b_{ij,k} \in \{0,1\}$ indicates whether these firms have also formed an R&D collaboration. We denote by $\mathbf{B} \in \{0,1\}^{nm}$ the $nm \times nm$ dimensional matrix with elements $b_{ij,k}$ and similarly by $\mathbf{A} \in \{0,1\}^{nm}$ the $nm \times nm$ dimensional matrix with elements $a_{ij,k}$.

Note that the profit function introduced in Equation (21) admits a potential function [cf. Monderer and Shapley, 1996], which not only accounts for quantity adjustments but also for the linking and market entry strategies, allowing firms to be active in several markets simultaneously.

**Proposition 4.** *The the profit function of Equation (21) admits a potential function* $\Phi\colon \mathbb{R}_{+}^{nm} \times$

$\{0,1\}^{nm} \times \{0,1\}^{nm} \to \mathbb{R}$ *given by*

$$\begin{aligned}\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) &= \sum_{i=1}^{n}\sum_{k=1}^{M}\mathbb{1}_{\{i\in\mathcal{M}_k\}}\left((\alpha_k-\bar{c}_{ik})q_{ik}-\frac{1}{2}q_{ik}^2-\gamma_{ik}\right)\\ &\quad -\frac{\rho}{2}\sum_{i=1}^{n}\sum_{k=1}^{M}\mathbb{1}_{\{i,j\in\mathcal{M}_k\}}\sum_{j\neq i}q_{ik}q_{jk}+\frac{\varphi}{2}\sum_{i=1}^{n}\sum_{k=1}^{M}\mathbb{1}_{\{i,j\in\mathcal{M}_k\}}\sum_{j=1}^{n}a_{ij}q_{ik}q_{jk}-\zeta m\\ &= \sum_{i=1}^{n}\sum_{k=1}^{M}\mathbb{1}_{\{i\in\mathcal{M}_k\}}\left((\alpha_k-\bar{c}_{ik})q_{ik}-\frac{1}{2}q_{ik}^2-\gamma_{ik}\right)\\ &\quad -\frac{\rho}{2}\sum_{i=1}^{n}\sum_{k=1}^{M}\sum_{j\neq i}b_{ij,k}q_{ik}q_{jk}+\frac{\varphi}{2}\sum_{i=1}^{n}\sum_{k=1}^{M}\sum_{j=1}^{n}a_{ij,k}q_{ik}q_{jk}-\zeta m,\end{aligned} \tag{22}$$

*where $m$ is the number of links in $G$.*

Similar to Proposition 2 with the potential function $\Phi(\mathbf{q},\mathbf{A},\mathbf{B})$ of Proposition 4 we are then able to characterize the stationary state of the stochastic process of quantity adjustment, market entry, exit, and link formation.

**Proposition 5.** *The dynamic process $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$ induces an irreducible and aperiodic Markov chain with a unique stationary distribution $\mu^\vartheta\colon \mathcal{Q}^n\times\{0,1\}^{n\times n}\times\mathcal{B}(n,m)\to[0,1]$ such that $\lim_{t\to\infty}\mathbb{P}(\boldsymbol{\omega}_t=(\mathbf{q},\mathbf{A},\mathbf{B})|\boldsymbol{\omega}_0=(\mathbf{q}_0,\mathbf{A}_0,\mathbf{B}_0))=\mu^\vartheta(\mathbf{q},\mathbf{A},\mathbf{B})$. When the shocks $\varepsilon$ are independent and identically exponentially distributed with parameter $\vartheta\geq 0$, then the probability measure $\mu^\vartheta$ is given by*

$$\mu^\vartheta(\mathbf{q},\mathbf{A},\mathbf{B})=\frac{e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B})-m\ln\left(\frac{\xi}{\lambda}\right))}}{\sum_{\mathbf{A}'\in\{0,1\}^{n\times n}}\sum_{\mathbf{B}'\in\mathcal{B}(n,m)}\sum_{\mathbf{q}'\in\mathcal{Q}^n}e^{\vartheta(\Phi(\mathbf{q}',\mathbf{A}',\mathbf{B}')-m'\ln\left(\frac{\xi}{\lambda}\right))}}, \tag{23}$$

*with the potential function $\Phi(\mathbf{q},\mathbf{A},\mathbf{B})$ given in Equation (22).*

## 6. Conclusion

In this paper we have introduced a model of multimarket, multiproduct competition among R&D collaborating firms. Through the formation of collaborations firms can benefit from technology spillovers to reduce their production costs (process innovations), while product innovations allow firms to enter novel markets with potentially fewer competitors. We derive a complete equilibrium characterization of the economy in which both, the market as well as the network structure are endogenously determined.

The paper could be extended along several dimensions. Similar to Hsie et al. [2012], one could estimate the model using maximum likelihood methods based on the equilibrium characterization of the model. These estimates could then be used to investigate the empirical differences in the R&D network structures observed across different industries [Rosenkopf and Schilling, 2007]. Moreover, similar to Ballester et al. [2006]; Hsie et al. [2012] one could perform a dynamic key player analysis considering the long run implications of firm exit when both, the network as well as the market structure react to the exit of a firm.

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# Appendix

## A. Bonacich Centrality

We introduce a network measure capturing the centrality of a firm in the network due to Bonacich [1987]. Let $\mathbf{A}$ be the symmetric $n \times n$ *adjacency matrix* of the network $G$ and $\lambda_{\mathrm{PF}}$ its largest real eigenvalue. The matrix $\mathbf{M}(G, \phi) = (\mathbf{I} - \phi\mathbf{A})^{-1}$ exists and is non-negative if and only if $\phi < 1/\lambda_{\mathrm{PF}}$.[9] Then

$$\mathbf{M}(G, \phi) = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k. \tag{24}$$

The Bonacich centrality vector is given by

$$\mathbf{b}_{\mathbf{u}}(G, \phi) = \mathbf{M}(G, \phi) \cdot \mathbf{u}, \tag{25}$$

where $\mathbf{u} = (1, \ldots, 1)^{\top}$. We can write the Bonacich centrality vector as

$$\mathbf{b}_{\mathbf{u}}(G, \phi) = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k \cdot \mathbf{u} = (\mathbf{I} - \phi\mathbf{A})^{-1} \cdot \mathbf{u}.$$

For the components $b_{\mathbf{u},i}(G, \phi)$, $i = 1, \ldots, n$, we get

$$b_{\mathbf{u},i}(G, \phi) = \sum_{k=0}^{\infty} \phi^k (\mathbf{A}^k \cdot \mathbf{u})_i = \sum_{k=0}^{\infty} \phi^k \sum_{j=1}^{n} \left(\mathbf{A}^k\right)_{ij}. \tag{26}$$

Because $\sum_{j=1}^{n} \left(\mathbf{A}^k\right)_{ij}$ counts the number of all walks of length $k$ in $G$ starting from $i$, $b_{\mathbf{u},i}(G, \phi)$ is the number of all walks in $G$ starting from $i$, where the walks of length $k$ are weighted by their geometrically decaying factor $\phi^k$.

## B. Proofs

*Proof of Proposition 1.* The potential $\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B})$ of Equation (8), which is given by

$$\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} (\eta q_i - \nu q_i^2) - \frac{\rho}{2} \sum_{i=1}^{n} \sum_{j \neq i} q_i q_j + \frac{\varphi}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_i q_j - \zeta m - \sum_{i=1}^{n} \sum_{m=1}^{M} \gamma^m \mathbb{1}_{\{i \in \mathcal{M}_m\}},$$

has the property that

$$\Phi(\mathbf{q}, \mathbf{A} + ij, \mathbf{B}) - \Phi(\mathbf{q}, \mathbf{A}, \mathbf{B}) = \varphi q_i q_j - \zeta = \pi_i(\mathbf{q}, \mathbf{A} + ij, \mathbf{B}) - \pi_i(\mathbf{q}, \mathbf{A}, \mathbf{B}). \tag{27}$$

The case of link removal is analogous. From the properties of $\pi_i(\mathbf{q}, \mathbf{A}, \mathbf{B})$ it also follows that $\Phi(q_i', \mathbf{q}_{-i}, \mathbf{A}, \mathbf{B}) - \Phi(q_i, \mathbf{q}_{-i}, \mathbf{A}, \mathbf{B}) = \pi_i(q_i', \mathbf{q}_{-i}, \mathbf{A}, \mathbf{B}) - \pi_i(q_i, \mathbf{q}_{-i}, \mathbf{A}, \mathbf{B})$. Next, consider product innovation. Recall that $b_{ij} \in \{0, 1\}$ indicates whether firms $i$ and $j$ are competitors in the same product market. Consider a change of firm $i$ leaving market $m$ and entering market $m'$. Let the associated competition matrices be given by $\mathbf{B} = (b_{ij})_{1 \leq i,j, \leq n}$ and $\mathbf{B}' = (b_{ij}')_{1 \leq i,j, \leq n}$, respectively. Note that the matrix $\mathbf{B}$ can always be represented in block diagonal form. We denote by $\mathcal{B}(n, m)$ the class of zero-one block diagonal $n \times n$ matrices with $m$ blocks and zero

[9] The proof can be found e.g. in Debreu and Herstein [1953].

diagonal, so that $\mathbf{B} \in \mathcal{B}(n,m)$. Furthermore, note that the matrix $\mathbf{B}'$ is obtained from the matrix $\mathbf{B}$ by a simultaneous row and column exchange, so that also the matrix $\mathbf{B}'$ can always be represented in block-diagonal form, that is $\mathbf{B}' \in \mathcal{B}(n,m)$. The marginal change in the profit of firm $i$ from leaving market $m$ and entering market $m'$ is given by

$$\begin{aligned}\pi_i(\mathbf{q},\mathbf{A},\mathbf{B}') - \pi_i(\mathbf{q},\mathbf{A},\mathbf{B}) &= \rho q_i \left( \sum_{j \in \mathcal{M}_{m'} \setminus \{i\}} q_j - \sum_{j \in \mathcal{M}_m \setminus \{i\}} q_j \right) - (\gamma^m - \gamma^{m'}) \\ &= \rho q_i \sum_{j=1}^n (b_{ij} - b'_{ij}) q_j - (\gamma^m - \gamma^{m'}).\end{aligned}$$

Note that for the potential function we get

$$\Phi(\mathbf{q},\mathbf{A},\mathbf{B}') - \Phi(\mathbf{q},\mathbf{A},\mathbf{B}) = \rho q_i \sum_{j=1}^n (b'_{ij} - b_{ij}) q_j - (\gamma^m - \gamma^{m'}) = \pi_i(\mathbf{q},\mathbf{A},\mathbf{B}') - \pi_i(\mathbf{q},\mathbf{A},\mathbf{B}).$$

This completes the proof. □

*Proof of Proposition 2.* In the following we show that the stationary distribution $\mu^{\vartheta}(\boldsymbol{\omega})$ in Equation (23) satisfies the detailed balance condition

$$\mu^{\vartheta}(\boldsymbol{\omega}) p(\boldsymbol{\omega}'|\boldsymbol{\omega}) = \mu^{\vartheta}(\boldsymbol{\omega}') p(\boldsymbol{\omega}|\boldsymbol{\omega}') \tag{28}$$

where $p(\boldsymbol{\omega}'|\boldsymbol{\omega})$ denotes the transition rate of the Markov chain from state $\boldsymbol{\omega}$ to $\boldsymbol{\omega}'$. Observe that the detailed balance condition is trivially satisfied if $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}$ differ in more than one link or more than one quantity level. Hence, we consider only the case of link creation $\mathbf{A}' = \mathbf{A} + ij$ (and removal $\mathbf{A}' = \mathbf{A} - ij$), an adjustment in quantity $q'_i \neq q_i$ for some $i \in \mathcal{N}$ or a change in the competition matrix $\mathbf{B}$ to $\mathbf{B}'$. For the case of link creation with a transition from $\boldsymbol{\omega} = (\mathbf{q},\mathbf{A},\mathbf{B})$ to $\boldsymbol{\omega}' = (\mathbf{q},\mathbf{A}+ij,\mathbf{B})$ we can write the detailed balance condition as follows

$$\begin{aligned}&\frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) - m \ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A}+ij,\mathbf{B})}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A}+ij,\mathbf{B})} + e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}} \lambda \\ &\qquad = \frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A}+ij,\mathbf{B}) - (m+1) \ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})} + e^{\vartheta\Phi(\mathbf{q},\mathbf{A}+ij,\mathbf{B})}} \xi.\end{aligned}$$

This equality is trivially satisfied. A similar argument holds for the removal of a link with a transition from $\boldsymbol{\omega} = (\mathbf{q},\mathbf{A},\mathbf{B})$ to $\boldsymbol{\omega}' = (\mathbf{q},\mathbf{A}-ij,\mathbf{B})$ where the detailed balance condition reads

$$\begin{aligned}&\frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) - m \ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A}-ij,\mathbf{B})}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A}-ij,\mathbf{B})} + e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}} \xi \\ &\qquad = \frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A}+ij,\mathbf{B}) - (m-1) \ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})} + e^{\vartheta\Phi(\mathbf{q},\mathbf{A}-ij,\mathbf{B})}} \lambda.\end{aligned}$$

For a change in the output level with a transition from $\boldsymbol{\omega} = (q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})$ to $\boldsymbol{\omega}' = (q'_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})$

we get for the detailed balance condition

$$\frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})-m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\pi_i(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B})}dq}{\int_{\mathcal{Q}} e^{\vartheta\pi_i(q',\mathbf{q}_{-i},\mathbf{A},\mathbf{B})}dq'}\chi$$
$$= \frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B})-m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\pi_i(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})}dq}{\int_{\mathcal{Q}} e^{\vartheta\pi_i(q',\mathbf{q}_{-i},\mathbf{A},\mathbf{B})}dq'}\chi.$$

This can be written as $e^{\vartheta(\Phi(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})-\Phi(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B}))} = e^{\vartheta(\pi_i(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B})-\pi_i(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B}))}$, which is satisfied since we have for the potential $\Phi(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B}) - \Phi(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B}) = \pi_i(q_i,\mathbf{q}_{-i},\mathbf{A},\mathbf{B}) - \pi_i(q_i',\mathbf{q}_{-i},\mathbf{A},\mathbf{B})$. Further, for a transition from $\mathbf{B} \in \mathcal{B}(n,m)$ to $\mathbf{B}' \in \mathcal{B}(n,m)$ we have that

$$\frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B})-m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B}')}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B}')}+e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}}\varkappa$$
$$= \frac{1}{\mathscr{Z}_\theta} e^{\vartheta(\Phi(\mathbf{q},\mathbf{A},\mathbf{B}')-m\ln\left(\frac{\xi}{\lambda}\right))} \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}}{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}+e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B}')}}\varkappa.$$

Hence, the probability measure $\mu^\vartheta(\boldsymbol{\omega})$ satisfies a detailed balance condition and therefore is the stationary distribution of the Markov chain with transition rates $p(\boldsymbol{\omega}'|\boldsymbol{\omega})$. $\square$

*Proof of Proposition 3.* We first prove part (i) of the proposition. Observe that the potential can be written as

$$\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) = \underbrace{\sum_{i=1}^{n}(\eta-\nu q_i)\,q_i}_{\psi(\mathbf{q})} - \sum_{i=1}^{n}\sum_{j=i+1}^{n} b_{ij}\rho q_i q_j + \sum_{i=1}^{n}\sum_{j=i+1}^{n} a_{ij}\underbrace{(\varphi q_i q_j - \zeta)}_{\sigma_{ij}}. \tag{29}$$

We then have that
$$e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})} = e^{\vartheta\psi(\mathbf{q})}e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}e^{\vartheta\sum_{i<j}^{n} a_{ij}\sigma_{ij}},$$

where only the second factor in the above expression is network dependent. We then can use the fact that

$$\sum_{\mathbf{A}\in\{0,1\}^{n\times n}} e^{\vartheta\sum_{i<j}^{n} a_{ij}\sigma_{ij}} = \prod_{i<j}\left(1+e^{\vartheta\sigma_{ij}}\right),$$

to obtain

$$\begin{aligned}
\sum_{\mathbf{A}\in\{0,1\}^{n\times n}} e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})} &= e^{\vartheta\psi(\mathbf{q})}e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}\sum_{\mathbf{A}\in\{0,1\}^{n\times n}} e^{\vartheta\sum_{i<j}^{n} a_{ij}\sigma_{ij}} \\
&= e^{\vartheta\psi(\mathbf{q})}e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}\prod_{i<j}\left(1+e^{\vartheta\sigma_{ij}}\right) \\
&= e^{\vartheta\psi(\mathbf{q})}e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}\prod_{i<j}\left(1+e^{\vartheta\sigma_{ij}}\right) \\
&= \prod_{i=1}^{n} e^{\vartheta(\eta-\nu q_i)q_i}e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}\prod_{i<j}\left(1+e^{\vartheta\varphi q_i q_j-\zeta}\right).
\end{aligned} \tag{30}$$

We can use Equation (30) to compute the marginal distribution

$$\mu^{\vartheta}(\mathbf{q},\mathbf{B}) = \frac{1}{\mathscr{Z}_{\vartheta}} \sum_{\mathbf{A}\in\{0,1\}^{n\times n}} e^{\vartheta\Phi(\mathbf{q},G)}$$
$$= \frac{1}{\mathscr{Z}_n^{\vartheta}} \prod_{i=1}^{n} e^{\vartheta(\eta-\nu q_i)q_i} e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j} \prod_{i<j}\left(1+e^{\vartheta(\varphi q_i q_j-\zeta)}\right). \tag{31}$$

With the marginal distribution from Equation (31) we can write the conditional distribution as

$$\begin{aligned}
\mu^{\vartheta}(\mathbf{A}|\mathbf{q},\mathbf{B}) = \frac{\mu^{\vartheta}(\mathbf{q},\mathbf{A},\mathbf{B})}{\mu^{\vartheta}(\mathbf{q},\mathbf{B})} &= \frac{e^{\vartheta\Phi(\mathbf{q},\mathbf{A},\mathbf{B})}}{\prod_{i=1}^{n} e^{\vartheta(\eta-\nu q_i)q_i} e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j} \prod_{i<j}\left(1+e^{\vartheta(\varphi q_i q_j-\zeta)}\right)} \\
&= \frac{e^{\vartheta\psi(\mathbf{q})} e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j} e^{\vartheta\sum_{i<j}^{n} a_{ij}\sigma_{ij}}}{\prod_{i=1}^{n} e^{\vartheta(\eta-\nu q_i)q_i} e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j} \prod_{i<j}\left(1+e^{\vartheta(\varphi q_i q_j-\zeta)}\right)} \\
&= \frac{e^{\vartheta\sum_{i<j}^{n} a_{ij}(\varphi q_i q_j-\zeta)}}{\prod_{i<j}\left(1+e^{\vartheta(\varphi q_i q_j-\zeta)}\right)} \\
&= \prod_{i<j} \frac{e^{\vartheta a_{ij}(\varphi q_i q_j-\zeta)}}{1+e^{\vartheta(\varphi q_i q_j-\zeta)}} \\
&= \prod_{i<j} \left(\frac{e^{\vartheta(\rho q_i q_j-\zeta)}}{1+e^{\vartheta(\rho q_i q_j-\zeta)}}\right)^{a_{ij}} \left(1-\frac{e^{\vartheta(\rho q_i q_j-\zeta)}}{1+e^{\vartheta(\rho q_i q_j-\zeta)}}\right)^{1-a_{ij}} \\
&= \prod_{i<j} p_{ij}^{a_{ij}} (1-p_{ij})^{1-a_{ij}}.
\end{aligned}$$

Hence, we obtain the likelihood of an inhomogeneous random graph for the collaboration network $G$, respectively, $\mathbf{A}$, with link probability

$$p_{ij} = \frac{e^{\vartheta(\rho q_i q_j-\zeta)}}{1+e^{\vartheta(\rho q_i q_j-\zeta)}}, \tag{32}$$

We next show that the networks $G$ in the support of the stationary distribution $\mu^{\vartheta}(\mathbf{q},\mathbf{A},\mathbf{B})$ in the limit of vanishing noise $\vartheta\to\infty$ are nested split graphs. A graph $G$ is a nested split graph if for every node $i\in\mathcal{N}$ there exist a weight $w_i$ and a threshold $\tau$ such that vertices $i$ and $j$ are linked if and only if $w_i+w_j\geq\tau$ [Mahadev and Peled, 1995]. In the following we denote by $\mathcal{H}^n\subset\mathcal{G}^n$ the set of nested split graphs with $n$ nodes.

Next, we consider part (ii) of the proposition. In the limit $\vartheta\to\infty$ the linking probability in Equation (32) becomes $\lim_{\vartheta\to\infty} p^{\vartheta}(q_i,q_j) = \mathbb{1}_{\{\rho q_i q_j>\zeta\}}$, and the conditional probability of the network $G$ can be written as $\lim_{\vartheta\to\infty}\mu^{\vartheta}(\mathbf{A}|\mathbf{q},\mathbf{B}) = \prod_{i<j}^{n}\mathbb{1}_{\{\rho q_i q_j>\zeta\}}^{a_{ij}}\mathbb{1}_{\{\rho q_i q_j<\zeta\}}^{1-a_{ij}}$. Assume that $G$ is a stochastically stable network, that is, we must have that $\lim_{\vartheta\to\infty}\mu^{\vartheta}(\mathbf{q},\mathbf{A},\mathbf{B})>0$. Since, $\mu^{\vartheta}(\mathbf{q},\mathbf{A},\mathbf{B}) = \mu^{\vartheta}(\mathbf{A}|\mathbf{q},\mathbf{B})\mu^{\vartheta}(\mathbf{q},\mathbf{B})$ this implies that $\lim_{\vartheta\to\infty}\mu^{\vartheta}(\mathbf{A}|\mathbf{q},\mathbf{B})>0$. It follows that $\rho q_i q_j>\zeta$ for all $a_{ij}=1$ and $\rho q_i q_j<\zeta$ for all $a_{ij}=0$. We then define the weights $w_i\equiv\log q_i$, $w_j\equiv\log q_j$ and a threshold $\tau\equiv\log\zeta-\log\rho$, and conclude that $G\in\mathcal{H}^n$ is a nested split graph.

Finally, we prove part (iii) of the proposition. We next consider the part of the stationary distribution $\mu^{\vartheta}(\mathbf{q},\mathbf{A},\mathbf{B})$ that depends on the market structure, that is the part that is proportional to $e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q_i q_j}$. Since all firms are identical, and due to symmetry, the stochastically stable state (when $\vartheta\to\infty$) will feature an identical output level $q$ across firms, and we can write this as $e^{-\vartheta\sum_{i<j}^{n} b_{ij}\rho q^2} = e^{-\vartheta\rho q^2\sum_{m=1}^{M}\binom{|\mathcal{M}_m|}{2}} = e^{-\vartheta\rho q^2\binom{n}{2}\sum_{m=1}^{M}x_m}$, where $|\mathcal{M}_m|$ is the number of firms

in sector $m$ and we have denoted by $x_m \equiv \frac{1}{\binom{n}{2}}\binom{|\mathcal{M}_m|}{2} = \frac{|\mathcal{M}_m|(|\mathcal{M}_m|-1)}{n(n-1)} \approx \left(\frac{|\mathcal{M}_m|}{n}\right)^2 = y_m^2$ and $y_m$ is the fraction of firms in market $\mathcal{M}_m$. Hence, we can write $e^{-\vartheta \sum_{i<j}^n b_{ij}\rho q^2} = e^{-c\sum_{m=1}^M y_m^2} = e^{-c\pi(\mathbf{y})^{-1}}$ with some constant $c$, and $\pi(\mathbf{y})$ is the participation ratio. The participation ratio $\pi(\mathbf{y})$ measures the number of elements of $\mathbf{y}$ which are dominant. We have that $1 \leq \pi(\mathbf{y}) \leq n$, where a value of $\pi(\mathbf{y}) = n$ corresponds to a fully homogenous case, while $\pi(\mathbf{y}) = 1$ corresponds to a fully concentrated case. As the stochastically stable states maximize the potential, this implies that in the stochastically stable state we have a fully homogeneous market structure in which the number of firms across sectors is equalized. $\square$

*Proof of Proposition 4.* The potential $\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B})$ of Equation (22), given by

$$\begin{aligned}
\Phi(\mathbf{q}, \mathbf{A}, \mathbf{B}) &= \sum_{i=1}^n \sum_{k=1}^M \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - \frac{1}{2}q_{ik}^2 - \gamma_{ik} \right) \\
&\quad - \frac{\rho}{2}\sum_{i=1}^n \sum_{k=1}^M \mathbb{1}_{\{i,j\in\mathcal{M}_k\}} \sum_{j\neq i} q_{ik}q_{jk} + \frac{\varphi}{2}\sum_{i=1}^n \sum_{k=1}^M \mathbb{1}_{\{i,j\in\mathcal{M}_k\}} \sum_{j=1}^n a_{ij}q_{ik}q_{jk} - \zeta m \\
&= \sum_{i=1}^n \sum_{k=1}^M \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - \frac{1}{2}q_{ik}^2 - \gamma_{ik} \right) \\
&\quad - \frac{\rho}{2}\sum_{i=1}^n \sum_{k=1}^M \sum_{j\neq i} b_{ij,k}q_{ik}q_{jk} + \frac{\varphi}{2}\sum_{i=1}^n \sum_{k=1}^M \sum_{j=1}^n a_{ij,k}q_{ik}q_{jk} - \zeta m,
\end{aligned}$$

has the property that

$$\Phi(\mathbf{q}, \mathbf{A}+ij, \mathbf{B}) - \Phi(\mathbf{q}, \mathbf{A}, \mathbf{B}) = \varphi \sum_{k=1}^M a_{ij,k}q_{ik}q_{jk} - \zeta = \pi_i(\mathbf{q}, \mathbf{A}+ij, \mathbf{B}) - \pi_i(\mathbf{q}, \mathbf{A}, \mathbf{B}), \tag{33}$$

where we have used the fact that the profit function in Equation (21) is given by

$$\pi_i = \sum_{k=1}^M \mathbb{1}_{\{i\in\mathcal{M}_k\}} \left( (\alpha_k - \bar{c}_{ik})q_{ik} - \frac{1}{2}q_{ik}^2 - \gamma_{ik} \right) - \rho \sum_{k=1}^M \sum_{j\neq i}^n b_{ij,k}q_{ik}q_{jk} + \varphi \sum_{k=1}^M \sum_{j=1}^n a_{ij,k}q_{ik}q_{jk} - \zeta_i.$$

The case of link removal is analogous. In the case of quantity adjustment (within the same sector), from the properties of $\pi_i(\mathbf{q}, \mathbf{A}, \mathbf{B})$ it also follows that $\Phi(q'_{ik}, \mathbf{q}_{-ik}, \mathbf{A}, \mathbf{B}) - \Phi(q_{ik}, \mathbf{q}_{-ik}, \mathbf{A}, \mathbf{B}) = \pi_i(q'_{ik}, \mathbf{q}_{-ik}, \mathbf{A}, \mathbf{B}) - \pi_i(q_{ik}, \mathbf{q}_{-ik}, \mathbf{A}, \mathbf{B})$.

Next, consider a product innovation. Recall that $b_{ij,k} \in \{0, 1\}$ indicates whether firms $i$ and $j$ are competitors in the same product market $\mathcal{M}_k$. Consider a change of firm $i$ leaving market $m$ and entering market $m'$. Let the associated competition matrices be given by $\mathbf{B} = (b_{ij,k})_{1\leq i,j,\leq n, 1\leq k\leq M}$ and $\mathbf{B}' = (b'_{ij,k})_{1\leq i,j,\leq n, 1\leq k\leq M}$, respectively. Note that the matrix $\mathbf{B}'$ is obtained from the matrix $\mathbf{B}$ by a simultaneous row and column exchange. The marginal change in the profit of firm $i$ from leaving market $m$ and entering market $m'$ is given by

$$\begin{aligned}\pi_i(\mathbf{q},\mathbf{A},\mathbf{B}')-\pi_i(\mathbf{q},\mathbf{A},\mathbf{B}) &= \rho q_{im}\left(\sum_{j\in\mathcal{M}_{m'}\setminus\{i\}} q_{jm'} - \sum_{j\in\mathcal{M}_{m}\setminus\{i\}} q_{jm}\right) - (\gamma^m - \gamma^{m'}) \\ &+ \varphi q_{im}\left(\sum_{j\in\mathcal{M}_{m'}\setminus\{i\}} a_{ij}q_{jm'} - \sum_{j\in\mathcal{M}_{m}\setminus\{i\}} a_{ij}q_{jm}\right) \\ &= \rho q_{im}\sum_{j=1}^{n}\sum_{k=1}^{M}(b_{ij,k}-b'_{ij,k})q_{jk} - (\gamma^m - \gamma^{m'}) \\ &+ \varphi q_{im}\sum_{j=1}^{n}\sum_{k=1}^{M}a_{ij}(b_{ij,k}-b'_{ij,k})q_{jk}.\end{aligned}$$

Note that for the potential function we get

$$\begin{aligned}\Phi(\mathbf{q},\mathbf{A},\mathbf{B}')-\Phi(\mathbf{q},\mathbf{A},\mathbf{B}) &= \rho q_{im}\sum_{j=1}^{n}\sum_{k=1}^{M}(b'_{ij,k}-b_{ij,k})q_{jk} - (\gamma^m - \gamma^{m'}) + \varphi q_{im}\sum_{j=1}^{n}\sum_{k=1}^{M}a_{ij}(b'_{ij,k}-b_{ij,k})q_{jk} \\ &= \pi_i(\mathbf{q},\mathbf{A},\mathbf{B}')-\pi_i(\mathbf{q},\mathbf{A},\mathbf{B}).\end{aligned}$$

This completes the proof. $\square$